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Note on the Turán number of the 3-linear hypergraph C_{13}

(https://arxiv.org/abs/2109.10520)

Introduction

The crown C_{13} is the linear 3-graph with vertices $\{a, b, c, d, e, f, g, h, i\}$ and edges

$$\{a, b, c\}, \{a, d, e\}, \{b, f, g\}, \{c, h, i\}.$$

We will prove the following conjecture of Gyárfás et al.: Any C_{13}-free linear 3-graph G satisfies

$$|E(G)| \leq \frac{3(n-s)}{2}$$

where s is the number of vertices with degree at least 6. This combined with previous work essentially determines Turán number for linear 3-graphs with at most 4 edges.

We will first summarize previous work, then state and proof two similar theorems on upper bounds of $E(G)$ where G is a crown-free graph.

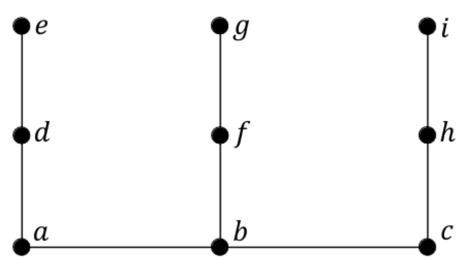

Figure 1: C_{13} as shown in the presented paper.

Important definitions

Definition 1 (Linear 3-graph) *A 3-graph $G = (V, E)$ consists of a finite set of vertices $V(G)$ and set of edges $E(G)$ where edges are 3-element subsets of V. A hypergraph is linear if any two edges share at most 1 vertex.*

Definition 2 (linear Turán number) *Linear Turán number $ex(n, F)$ is the maximum number of edges in any F-free linear 3-graph on n vertices.*

Definition 3 (F-free graph) *A graph is F-free if it does not contain F as a subgraph.*

Definition 4 (Minimal counter-example) *A minimal counter-example is a counter-example that contains no proper subgraph that is also a counter-example.*

Previous work

Gyárfás, Ruszinkó and Sárközy showed

$$6\left\lfloor\frac{n-3}{4}\right\rfloor + \varepsilon \leq ex(n, C_{13}) \leq 2n$$

where ε changes based on the nominator. It is 0 whenever $n-3 \equiv 0,1 \mod 4$, it is 1 whenever $n-3 \equiv 2 \mod 4$ and 3 otherwise.

In a different paper Gyárfás et. al. showed that every linear 3-graph with minimum degree 4 is not crown free.

Recently, Fletcher improved the previous upper bound to

$$ex(n, C_{13}) \leq \frac{5}{3}n.$$

Stronger bounds

In this talk, we will prove the following two upper bounds and try to connect them to the previous results.

Theorem 5 *Let G be any crown-free linear 3-graph G on n vertices. Then its number of edges satisfies*

$$|E(G)| \leq \frac{3(n-s)}{2}$$

where s is the number of vertices in G with degree at least 6.

Theorem 6 *Let G be any crown-free linear 3-graph G on n vertices, and let s be the number of vertices in G with degree at least 6. If $s \leq 2$, then the number of edges satisfies*

$$|E(G)| \leq \frac{10(n-s)}{7}.$$

Outline of proofs of the theorems

We will use the following notation $D(\{x,y,z\}) \geq \langle a,b,c \rangle$ to say that $d(x) \geq a$, $d(y) \geq b$ and $d(z) \geq c$ for positive integers a,b,c where $a \geq b \geq c$.

To do the proofs we also need the lemma stated below.

Lemma 7 *Let G be a crown-free graph and $e = \{x,y,z\} \in E(G)$ satisfy $D(e) \geq \langle 5,5,4 \rangle$. Then, the vertex set of all vertices sharing an edge with $\{x,y,z\}$*

$$S = \bigcup_{f \in E(G), f \cap \{x,y,z\} \neq \emptyset} f,$$

contains exactly 11 vertices and all vertices in S have degree at most 5. The set of edges that contain at least one vertex in S,

$$E_S = \{f : f \in E(G), f \cap S \neq \emptyset\}$$

contains at most 13 edges, and all elements of E_S are subsets of S.

Both theorems can be proved by contradiction. We assume a minimal counter-example. We always start by showing the existence of some special edge based on some equality on the degrees of its vertices. We then analyze $D(e)$ of this particular edge. This yields multiple cases. With most of them, it is simple to show that they lead to a contradiction. Only complicated case is the one when $D(e) \geq \langle 5,5,4 \rangle$. However, using the previous lemma we can show that the assumed minimal counterexample can be made even smaller.

Proofs of theorems are relatively straightforward as we will see. The hardest part is to prove the lemma which we might attempt at the end of the talk in case we have enough time left.

Bibliography

[1] András Gyárfás and Miklós Ruszinkó and Gábor N. Sárközy *Linear Turán numbers of acyclic triple systems.* European Journal of Combinatorics

[2] Willem Fletcher *Improved Upper Bound on the Linear Tur an Number of the Crown* arXiv:2109.02729

[3] Alvaro Carbonero and Willem Fletcher and Jing Guo and András Gyárfás and Rona Wang and Shiyu Yan *Crowns in linear* 3*-graphs* arXiv:2107.14713